

# A Predictor-Corrector Hybrid Method for Numerical Approximation of Third-Order Initial Value Problems

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## Abstract

For solving third order ordinary differential equations, this work discusses a numerical integrator with continuous coefficients. For the approximate solution of the differential equations, a combination of power series and exponential function has been employed as the basis function in the formulation of the integrators. To produce a system of linear equations, the approximation solution and the accompanying differential system were interpolated and collocated, respectively. The method developed is accurate to a high order, stable, and convergent, making it suited for the integration of stiff systems of initial value problems of odes. The idea of creating initial values with a lower order of accuracy than the main scheme was avoided in this study by providing the same order of accuracy as the main technique. The integration identities as equal areas under the various segments of the solution curves over the integration intervals brought to the natural retention of symmetry as a result of this novel idea. The application of this newly discovered multistep integration method to a number of well-known issues in the literature yields correct results at a cheap cost of computing. When compared to previous methods, the results appear to be more accurate.

**Keywords:** Predictor-Corrector method, Linear multistep, continuous scheme, Hybrid method, Third order, Power series and exponential function

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## 1. Introduction

Integration of third-order differential equations of the form

$$y'' = f(x, y, y', y''), \quad y(x_0) = \psi_0, \quad y'(x_0) = \psi_1, \quad y''(x_0) = \psi_2 \quad (1)$$

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where  $y \in C''[a, b]$  and  $f$  are continuous function of  $x, y$ , and derivatives of  $y$  is a topic of great interest in a variety of scientific fields, including fluid dynamics, electromagnetic waves, and gravity driven flow. The traditional approach of solving (1) is to reduce it to an analogous system of first-order system of initial value problems (ivps) of ordinary differential equations of the form

$$y' = f(x, y), \quad y(a) = \psi; \quad a \leq x \leq b, \quad x, y \in R^n, \quad f \in C'[a, b] \quad (2)$$

for any appropriate numerical method such as Euler's and Runge-Kutta methods to solve the resultant system reported in [1-5]. The method's downsides have been claimed to include computational burden, intricacy in designing computer programs and resulting computer time waste, and the method's inability to use additional information associated with a given ODE, such as the oscillatory nature of the solution [6]. The increasing dimension of the issue, as well as the low level of accuracy of the methods used to solve the system of first-order IVPs of ODEs, have resulted in these recognized drawbacks.

As a result of these difficulties, authors have recently attempted to formulate numerical algorithms capable of directly solving problem (1) using various methods of derivation with varying degrees of accuracy, such as [7-9].

The implicit LMM requires the establishment of initial values before it can be used to solve issues. The predictors are the algorithms that generate beginning values, while the corrector is the implicit linear multistep method itself. The Predictor-Corrector approach is the name given to the combination of the two when employed to produce a numerical solution to a specific problem.

This technique is repeated until all of the desired results are achieved. The beginning values for the scheme were then expanded using Taylor's series expansion.

[10, 11] presented algorithmic collocation and p-stable linear multistep methods for solving fourth and third order ODEs, respectively, in constructing a predictor to start the corrector in the predictor-corrector. [12] developed explicit LMMs by omitting collocation in the k-step technique, and the predictors' order is lower than the correctors' order. The use of lower order

predictors to perform the scheme published by [13] is the key disadvantage of this strategy as stated in the literature.

The development of an explicit linear multistep method whose order is the same as the order of the method (corrector) as this has a beneficial effect on the accuracy of the methods is a simple way to determine  $\{y_{n+i}^{(n)}\}$ , which is the predictor to the numerical solution of the  $n$ th order beginning value [14].

For the creation of a continuous hybrid linear multistep technique for direct solution of problem (1) in this work, a combination of power series and exponential function was employed as the basic function in creating the interpolation and collocation equations.

## 2. Materials and Methods

The development of continuous implicit four-point hybrid methods for the solution of IVPs of higher order ODEs is described in this section. The goal is to use a combination of power series and exponential functions of the type (3) to approximate the solution  $y(x)$  of equation (5) in the partition  $\pi_{[a,b]} = [a = x_0 < \dots < x_n = b]$  of the integration interval  $[a, b]$ .

$$y(x) = \sum_{j=0}^{c+i-1} a_j x^j + a_{c+i} \sum_{j=0}^{c+i} \frac{\alpha^j x^j}{j!} \quad (3)$$

The main function for the creation of the approach is Equation (3), where  $c$  and  $i$  are the number of collocation and interpolation points respectively,  $a_j$ 's are to be found.

Equation (3) generates a differential system, which is written as

$$y''(x) = \sum_{j=3}^{c+i-1} j(j-1)(j-2)a_j x^{j-3} + a_{c+i} \sum_{j=3}^{c+i} \frac{\alpha^j x^{j-3}}{(j-3)!} \quad (4)$$

Equations (3) interpolated at both grid and off-grid points,  $x = x_{n+i}$ ,  $i = 0(1/2)k - 1$  while equation (4) is collocated at all grid points,  $x = x_{n+i}$ ,  $i = 0(1)k$ ,  $x \in R$ , where  $R$  is the set of real numbers. The matrix equation (5) represents the system of equations generated from the aforementioned interpolation and collocation.

$$Ax = B \quad (5)$$

$$\begin{bmatrix}
 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5 & 504x_n^6 & [\alpha^3 + \alpha^4 x_n + \dots + \frac{\alpha^{10} x_n^7}{7!}] \\
 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5 & 504x_{n+1}^6 & [\alpha^3 + \alpha^4 x_{n+1} + \dots + \frac{\alpha^{10} x_{n+1}^7}{7!}] \\
 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & 336x_{n+2}^5 & 504x_{n+2}^6 & [\alpha^3 + \alpha^4 x_{n+2} + \dots + \frac{\alpha^{10} x_{n+2}^7}{7!}] \\
 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & 336x_{n+3}^5 & 504x_{n+3}^6 & [\alpha^3 + \alpha^4 x_{n+3} + \dots + \frac{\alpha^{10} x_{n+3}^7}{7!}] \\
 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & [1 + \alpha x_n + \frac{\alpha^2 x_n^2}{2!} + \dots + \frac{\alpha^{10} x_n^{10}}{10!}] \\
 1 & x_{n+r} & x_{n+r}^2 & x_{n+r}^3 & x_{n+r}^4 & x_{n+r}^5 & x_{n+r}^6 & x_{n+r}^7 & x_{n+r}^8 & x_{n+r}^9 & [1 + \alpha x_{n+r} + \frac{\alpha^2 x_{n+r}^2}{2!} + \dots + \frac{\alpha^{10} x_{n+r}^{10}}{10!}] \\
 1 & x_{n+s} & x_{n+s}^2 & x_{n+s}^3 & x_{n+s}^4 & x_{n+s}^5 & x_{n+s}^6 & x_{n+s}^7 & x_{n+s}^8 & x_{n+s}^9 & [1 + \alpha x_{n+s} + \frac{\alpha^2 x_{n+s}^2}{2!} + \dots + \frac{\alpha^{10} x_{n+s}^{10}}{10!}] \\
 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 & [1 + \alpha x_{n+1} + \frac{\alpha^2 x_{n+1}^2}{2!} + \dots + \frac{\alpha^{10} x_{n+1}^{10}}{10!}] \\
 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 & [1 + \alpha x_{n+2} + \frac{\alpha^2 x_{n+2}^2}{2!} + \dots + \frac{\alpha^{10} x_{n+2}^{10}}{10!}] \\
 1 & x_{n+u} & x_{n+u}^2 & x_{n+u}^3 & x_{n+u}^4 & x_{n+u}^5 & x_{n+u}^6 & x_{n+u}^7 & x_{n+u}^8 & x_{n+u}^9 & [1 + \alpha x_{n+u} + \frac{\alpha^2 x_{n+u}^2}{2!} + \dots + \frac{\alpha^{10} x_{n+u}^{10}}{10!}] \\
 1 & x_{n+v} & x_{n+v}^2 & x_{n+v}^3 & x_{n+v}^4 & x_{n+v}^5 & x_{n+v}^6 & x_{n+v}^7 & x_{n+v}^8 & x_{n+v}^9 & [1 + \alpha x_{n+v} + \frac{\alpha^2 x_{n+v}^2}{2!} + \dots + \frac{\alpha^{10} x_{n+v}^{10}}{10!}]
 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{bmatrix} = \begin{bmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ y_n \\ y_{n+u} \\ y_{n+v} \\ y_{n+1} \\ y_{n+2} \\ y_{n+r} \\ y_{n+s} \end{bmatrix}$$

Solving (5) for  $a_j$ 's and substituting their results into (2) using the transformation in

Obarhua and Adegboro [15],  $t = \frac{1}{h}(x - x_{n+k-1})$ ,  $\frac{dt}{dx} = \frac{1}{h}$  and simplifying the result gives a

continuous hybrid four-step method (3S4PHM)

$$y(t) = \sum_{j=0}^{k-1} \alpha_j(t) y_{n+j} + \{\tau_1(t) y_{n+r} + \tau_2(t) y_{n+s} + \tau_3(t) y_{n+u} + \tau_4(t) y_{n+v}\} + h^3 \sum_{j=0}^k \beta_j(t) f_{n+j} \quad (6)$$

Taking  $k = 3$ , the continuous coefficients  $\alpha_j$ 's,  $\beta_j$ 's,  $\tau_j$ 's and their respective first and second derivatives can be calculated as follows:

$$\alpha_0(t) = \left\{ -\frac{334611}{2754944}t^{10} - \frac{1673055}{2754944}t^9 - \frac{10021563}{19284608}t^8 + \frac{15091029}{9642304}t^7 + \frac{7026831}{2754944}t^6 - \frac{1037367}{2754944}t^5 - \right. \\ \left. \frac{5052699}{2754944}t^4 + \frac{2046805}{4821152}t^2 - \frac{97925}{1205288}t \right\}$$

$$\alpha_1(t) = \left\{ \frac{3035227033647}{3359198781320}t^{10} + \frac{787296360717}{167959939066}t^9 + \frac{14778754347861}{3359198781320}t^8 - \frac{4987185092406}{419899847665}t^7 - \right. \\ \left. \frac{8456675836833}{419899847665}t^6 + \frac{921924248364}{419899847665}t^5 + \frac{5038457699232}{419899847665}t^4 - \frac{482371985992}{419899847665}t^2 + \right. \\ \left. \frac{655528736}{83979969533}t \right\}$$

$$\tau_1(t) = \left\{ \frac{1340285954823}{1679599390660}t^{10} + \frac{689983320033}{1679599390660}t^9 + \frac{44590496925321}{11757195734620}t^8 - \frac{31001673238002}{2939298933655}t^7 - \right. \\ \left. \frac{7566844055121}{419899847665}t^6 + \frac{1011186898308}{419899847665}t^5 + \frac{5038457699232}{419899847665}t^4 - \frac{9028586979504}{2939298933655}t^2 + \right. \\ \left. \frac{1753076198592}{2939298933655}t \right\}$$

$$\tau_2(t) = \left\{ \frac{39783782166039}{26873590250560}t^{10} - \frac{41219629879743}{5374718050112}t^9 - \frac{192234397595121}{26873590250560}t^8 + \frac{264835394616159}{13436795125280}t^7 - \right. \\ \left. + \frac{901763606819691}{26873590250560}t^6 - \frac{122623307363907}{26873590250560}t^5 - \frac{624033002785591}{26873590250560}t^4 + \frac{32913101005029}{6718397562640}t^2 \right. \\ \left. - \frac{73670325552}{83979969533}t \right\}$$

$$\tau_3(t) = \left\{ \frac{412941486483}{3359198781320}t^{10} + \frac{74052806337}{83979969533}t^9 + \frac{48638951134739}{23514391469240}t^8 + \frac{2485005259256}{2939298933655}t^7 - \right. \\ \left. \frac{2119974235344}{419899847665}t^6 - \frac{4352720495832}{419899847665}t^5 - \frac{3358397181276}{419899847665}t^4 + \frac{12382035379272}{2939298933655}t^2 + \right. \\ \left. \frac{1198363629024}{587859786731}t \right\}$$

$$\tau_4(t) = \left\{ \begin{array}{l} -\frac{1954469720367}{53747180501120} t^{10} - \frac{3224251648143}{10749436100224} t^9 - \frac{316146947940567}{376230263507840} t^8 - \frac{92536492995207}{188115131753920} \\ t^7 + \frac{108927741542139}{53747180501120} t^6 + \frac{227955062217213}{53747180501120} t^5 + \frac{146289279781449}{53747180501120} t^4 - \frac{41114147498847}{94057565876960} \\ t^2 - \frac{69116481981}{4702878293848} t \end{array} \right\}$$

$$\alpha_2(t) = \left\{ \begin{array}{l} -\frac{5004884019033}{26873590250560} t^{10} - \frac{5916551292801}{5374718050112} t^9 - \frac{48099775873743}{26873590250560} t^8 + \frac{10696457023137}{13436795125280} t^7 + \\ \frac{136411676304309}{26873590250560} t^6 + \frac{173619955135683}{26873590250560} t^5 + \frac{141485001281847}{26873590250560} t^4 - \frac{32775723990193}{6718397562640} \\ t^2 - \frac{69116481981}{419899847665} t + 1 \end{array} \right\}$$

$$\beta_0(t) = \left\{ \begin{array}{l} -\frac{2353347693}{6718397562640} t^{10} - \frac{1077011595}{671839756264} t^9 - \frac{1563777871}{1343679512528} t^8 + \frac{1395356855}{335919878132} t^7 + \\ \frac{377867362609}{60465578063760} t^6 - \frac{720600587}{671839756264} t^5 - \frac{5375057575441}{12093115612752} t^4 + \frac{15134577607}{15116394515940} t^2 - \\ \frac{15661771}{83979969533} t \end{array} \right\}$$

$$\beta_1(t) = \left\{ \begin{array}{l} \frac{126162537039}{6718397562640} t^{10} + \frac{17262127377}{167959939066} t^9 + \frac{75150932811}{671839756264} t^8 - \frac{86366916157}{335919878132} t^7 - \\ \frac{10300161224719}{20155192687920} t^6 + \frac{45008396801}{1007759634396} t^5 + \frac{204072154045}{503879817198} t^4 - \frac{46408224041}{419899847665} t^2 + \\ \frac{5965219084}{251939908599} t \end{array} \right\}$$

$$\beta_2(t) = \left\{ \begin{array}{l} \frac{14069586951}{6718397562640} t^{10} + \frac{1625254308}{83979969533} t^9 + \frac{77106667533}{1343679512528} t^8 + \frac{4299655443}{167959939066} t^7 - \\ \frac{3406265308241}{20155192687920} t^6 - \frac{67728598114}{251939908599} t^5 - \frac{85467293263}{4031038537584} t^4 + \frac{1}{6} t^3 + \frac{225734694817}{5038798171980} \\ t^2 - \frac{2411603503}{83979969533} t \end{array} \right\}$$

$$\beta_3(t) = \left\{ \begin{array}{l} \frac{245298393}{6718397562640} t^{10} + \frac{27788931}{83979969533} t^9 + \frac{776908529}{671839756264} t^8 + \frac{614982655}{335919878132} t^7 + \\ \frac{54683191421}{60465578063760} t^6 - \frac{860445169}{1007759634396} t^5 - \frac{1471293169}{1511639451594} t^4 + \frac{633859037}{3779098628985} t^2 - \\ \frac{4064732}{251939908599} t \end{array} \right\} \quad (7)$$

The first and second derivatives of (36) are

$$\alpha'_0(t) = \left\{ \begin{array}{l} -\frac{1673055}{1377472} t^9 - \frac{15057495}{2754944} t^8 - \frac{10031563}{2410576} t^7 + \frac{15091029}{1377472} t^6 + \frac{21080493}{1377472} t^5 - \frac{5186835}{2754944} t^4 - \\ \frac{5052699}{688736} t^3 + \frac{2046805}{2410576} t - \frac{97925}{1205288} \end{array} \right\}$$

$$\alpha'_1(t) = \left\{ \begin{array}{l} \frac{3035227033647}{335919878132} t^9 + \frac{7085667246453}{167959939066} t^8 + \frac{14778754347861}{419899847665} t^7 - \frac{34910295646842}{419899847665} t^6 - \\ \frac{50740055020998}{419899847665} t^5 + \frac{921924248364}{83979969533} t^4 + \frac{20153830796928}{419899847665} t^3 - \frac{964743971984}{419899847665} t + \\ \frac{655528736}{83979969533} \end{array} \right\}$$

$$\alpha'_2(t) = \left\{ \begin{array}{l} -\frac{500488401903}{2687359025056} t^9 - \frac{53248961635209}{5374718050112} t^8 - \frac{48099775873743}{3359198781320} t^7 + \frac{74875199161959}{13436795125280} t^6 + \\ \frac{409235028912927}{13436795125280} t^5 + \frac{173619955135683}{5374718050112} t^4 + \frac{141485001281847}{6718397562640} t^3 - \frac{32775723990193}{3359198781320} t \\ - \frac{69116481981}{419899847665} \end{array} \right\}$$

$$\tau'_1(t) = \left\{ \begin{array}{l} \frac{690142977416}{83979969533} t^9 + \frac{6209849880297}{1679599390660} t^8 + \frac{89180993850643}{2939298933655} t^7 - \frac{31001673238002}{419899847665} t^6 - \\ \frac{45401064330726}{419899847665} t^5 + \frac{1011186898308}{83979969533} t^4 + \frac{20153830796928}{419899847665} t^3 - \frac{18057173959008}{2939298933655} t + \\ \frac{1753076198592}{2939298933655} \end{array} \right\}$$

$$\tau'_2(t) = \left\{ \begin{array}{l} \frac{39783782166039}{2687359025056} t^9 - \frac{370976668917687}{5374718050112} t^8 - \frac{192234397595121}{3359198781320} t^7 + \frac{1853847762313113}{13436795125280} t^6 \\ + \frac{2705290820459073}{13436795125280} t^5 - \frac{122623307363907}{5374718050112} t^4 - \frac{624023002785591}{6718397562640} t^3 + \frac{32913101005029}{3359198781320} t \\ - \frac{73670325552}{83979969533} \end{array} \right\}$$

$$\tau'_3(t) = \left\{ \begin{array}{l} \frac{412941486483}{335919878132} t^9 + \frac{666475257033}{83979969533} t^8 + \frac{49638951134739}{2939298933655} t^7 + \frac{2485005259356}{419899847665} t^6 - \\ \frac{12719845412064}{419899847665} t^5 - \frac{4352720495832}{83979969533} t^4 - \frac{13433588725104}{419899847665} t^3 + \frac{24764070758544}{2939298933655} t \\ + \frac{1198363629024}{587859786731} \end{array} \right\}$$

$$\tau'_4(t) = \left\{ \begin{array}{l} -\frac{1954469720367}{5374718050112} t^9 - \frac{29018264833287}{10749436100224} t^8 - \frac{316146947940567}{47028782938480} t^7 - \frac{92536492995207}{26873590250560} t^6 + \\ \frac{326783224626417}{26873590250560} t^5 + \frac{227955062217213}{10749436100224} t^4 + \frac{146289279781449}{13436795125280} t^3 - \frac{41114147498847}{47028782938480} t \\ - \frac{69116481981}{4702878293848} \end{array} \right\}$$

$$\beta'_0(t) = \left\{ \begin{array}{l} -\frac{23533476693}{671839756264} t^9 - \frac{9693104355}{671839756264} t^8 - \frac{1563777871}{167959939066} t^7 + \frac{9767497985}{335919878132} t^6 \\ - \frac{15661771}{83979969533} + \frac{377867362609}{10077596343960} t^5 - \frac{3603002935}{671839756264} t^4 - \frac{53750575441}{3023278903188} t^3 \\ + \frac{15134577607}{7558197257970} t \end{array} \right\}$$

$$\beta'_1(t) = \left\{ \begin{array}{l} \frac{126162537039}{671839756264} t^9 + \frac{155359146393}{167959939066} t^8 + \frac{75150932811}{83979969533} t^7 - \frac{604414413099}{335919878132} t^6 - \\ \frac{10300161224719}{3359198781320} t^5 - \frac{225041984005}{1007759634396} t^4 + \frac{408144308090}{251939908599} t^3 - \frac{2816448082}{419899847665} t \\ + \frac{5965219084}{251939908599} \end{array} \right\}$$

$$\begin{aligned}
 \beta'_2(t) &= \left\{ \frac{14069586951}{671839756264} t^9 + \frac{14627288772}{83979969533} t^8 + \frac{77106667533}{167959939066} t^7 + \frac{30097588101}{167959939066} t^6 - \right. \\
 &\quad \left. \frac{3406265308241}{3359198781320} t^5 - \frac{338642990570}{251939908599} t^4 - \frac{85467293263}{1007759634396} t^3 + \frac{1}{2} t^2 + \frac{225734694817}{2519399085990} t \right\} \\
 \beta'_3(t) &= \left\{ \frac{245298393}{671839756264} t^9 + \frac{250100379}{83979969533} t^8 + \frac{776908529}{83979969533} t^7 + \frac{4304878585}{335919878132} t^6 + \right. \\
 &\quad \left. \frac{54683191421}{10077596343960} t^5 - \frac{4302225845}{10077596343960} t^4 - \frac{2942586338}{755819725797} t^3 + \frac{1267718074}{3779098628985} t \right\} \quad (8) \\
 \alpha''_0(t) &= \left\{ \frac{15057495}{1377472} t^8 - \frac{15057495}{344368} t^7 - \frac{10031563}{344368} t^6 + \frac{45273087}{688736} t^5 + \frac{105402465}{1377472} t^4 \right\} \\
 &\quad \left. - \frac{5186835}{688736} t^4 - \frac{15158097}{688736} t^2 + \frac{2046805}{2410576} \right\} \\
 \alpha''_1(t) &= \left\{ \frac{27317043302823}{335919878132} t^8 + \frac{28342668985812}{83979969533} t^7 + \frac{103451280435027}{419899847665} t^6 \right. \\
 &\quad \left. - \frac{209461773881052}{419899847665} t^5 - \frac{50740055020998}{83979969533} t^4 + \frac{3687696993456}{83979969533} t^3 \right\} \\
 &\quad + \frac{60461492390784}{419899847665} t^2 - \frac{964743971984}{419899847665} \\
 \alpha''_2(t) &= \left\{ \frac{4504395617127}{2687359025056} t^8 - \frac{53248961635209}{671839756264} t^7 - \frac{336698431116201}{3359198781320} t^6 + \frac{224625597485877}{6718397562640} t^5 \right\} \\
 &\quad + \frac{409235028912927}{2687359025056} t^4 + \frac{173619955135683}{1343679512528} t^3 + \frac{424455003845541}{6718397562640} t^2 - \frac{32775723990193}{3359198781320} \\
 \tau''_1(t) &= \left\{ \frac{6031286796744}{83979969533} t^8 + \frac{12419699760594}{419899847665} t^7 + \frac{89180993850643}{419899847665} t^6 - \frac{186010039428012}{419899847665} t^5 \right\} \\
 &\quad - \frac{45401064330726}{83979969533} t^4 + \frac{3687696993456}{83979969533} t^3 + \frac{60461492390784}{419899847665} t^2 - \frac{18057173959008}{2939298933655}
 \end{aligned}$$

$$\begin{aligned}
 \tau_2''(t) &= \left\{ \frac{\frac{358054039494351}{2687359025056} t^8 - \frac{370976668917687}{671839756264} t^7 - \frac{1345640783165847}{3359198781320} t^6 + \frac{5561543286939339}{6718397562640} t^5 + \frac{2705290820459073}{2687359025056} t^4 - \frac{122623307363907}{1343679512528} t^3}{-\frac{1872069008356773}{6718397562640} t^2 + \frac{32913101005029}{3359198781320}} \right\} \\
 \tau_3''(t) &= \left\{ \frac{\frac{3716473378347}{335919878132} t^8 + \frac{5331802056264}{83979969533} t^7 + \frac{49638951134739}{419899847665} t^6 + \frac{14910031556136}{419899847665} t^5 - \frac{12719845412064}{83979969533} t^4 - \frac{17410881983328}{83979969533} t^3 - \frac{40300766175312}{419899847665} t^2 + \frac{24764070758544}{2939298933655}} \right\} \\
 \tau_4''(t) &= \left\{ -\frac{\frac{17590227483303}{5374718050112} t^8 - \frac{29018264833287}{1343679512528} t^7 - \frac{316146947940567}{6718397562640} t^6 - \frac{277609478985621}{13436795125280} t^5}{+\frac{326783224626417}{5374718050112} t^4 + \frac{227955062217213}{2687359025056} t^3 + \frac{438867839344347}{13436795125280} t^2 - \frac{41114147498847}{47028782938480}} \right\} \\
 \beta_0''(t) &= \left\{ -\frac{\frac{21180129237}{671839756264} t^8 - \frac{9693104355}{83979969533} t^7 - \frac{10946445097}{167959939066} t^6 + \frac{29302493955}{167959939066} t^5}{\frac{377867362609}{2015519268792} t^4 - \frac{3603002935}{167959939066} t^3 - \frac{53750575441}{1007759634396} t^2 + \frac{15134577607}{7558197257970}} \right\} \\
 \beta_1''(t) &= \left\{ \frac{\frac{1135462833351}{671839756264} t^8 + \frac{671436585572}{83979969533} t^7 + \frac{526056529677}{83979969533} t^6 - \frac{1813243239297}{167959939066} t^5}{-\frac{10300161224719}{671839756264} t^4 - \frac{225041984005}{251939908599} t^3 + \frac{408144308090}{83979969533} t^2 - \frac{92816448082}{419899847665}} \right\} \\
 \beta_2''(t) &= \left\{ \frac{\frac{126626282559}{671839756264} t^8 + \frac{117018310176}{83979969533} t^7 + \frac{539746672731}{167959939066} t^6 + \frac{90292764303}{83979969533} t^5}{\frac{3406265308241}{671839756264} t^4 - \frac{1354571962280}{251939908599} t^3 - \frac{85467293263}{335919878132} t^2 + t + \frac{225734694817}{2519399085990}} \right\} \\
 \beta_3''(t) &= \left\{ \frac{\frac{2207685537}{671839756264} t^8 + \frac{2000803032}{83979969533} t^7 + \frac{5438359703}{83979969533} t^6 + \frac{12914635755}{167959939066} t^5}{\frac{54683191421}{2015519268792} t^4 - \frac{4302225845}{251939908599} t^3 - \frac{2942586338}{251939908599} t^2 + \frac{1267718074}{3779098628985}} \right\} \quad (9)
 \end{aligned}$$

The values of  $t$  could be taken anywhere in the interval  $I = (0, 1]$  to get a discrete scheme, however in this paper, we use  $t = 1$  in (6) to get a zero-stable method and its derivatives.

$$y_{n+3} = \frac{1}{19509355} \left[ \begin{array}{l} 19509355y_n - 134017173y_{\frac{n+1}{3}} + 257536746y_{\frac{n+2}{3}} - 175049238y_{n+1} + 175049238y_{n+2} \\ - 257536746y_{\frac{n+7}{3}} + 134017173y_{\frac{n+8}{3}} \end{array} \right] + \frac{h^3}{3901871} [10080f_n - 670544f_{n+1} - 670544f_{n+2} + 10080f_{n+3}] \quad (10)$$

$$hy'_{n+3} = \frac{1}{11757195734620} \left[ \begin{array}{l} \frac{163912469749635}{2}y_n - 563168752343496y_{\frac{n+1}{3}} + 1075463626315869y_{\frac{n+2}{3}} - \\ 715719443637625y_{n+1} + 609916645162531y_{n+2} - 842758029830079y_{\frac{n+7}{3}} + \\ \frac{708619438915965}{2}y_{\frac{n+8}{3}} \end{array} \right] + \frac{h^2}{3901871} [4536381116f_n - 305340277940f_{n+1} - 264206753858f_{n+2} + 5792783578f_{n+3}] \quad (11)$$

$$h^2 y''_{n+3} = \frac{1}{9405756587696} \left[ \begin{array}{l} 280332415290569y_n - 1926285046201080y_{\frac{n+1}{3}} + 3652918696946862y_{\frac{n+2}{3}} \\ - 2372515848804556y_{n+1} + 1618708185218642y_{n+2} - 2051013307012668y_{\frac{n+7}{3}} \\ + 797854904562231y_{\frac{n+8}{3}} \end{array} \right] + h \left[ \begin{array}{l} \frac{291119546126}{3779098628985}f_n - \frac{13203623705357}{2519399085990}f_{n+1} - \frac{4712352281219}{1259699542995}f_{n+2} + \frac{1265772755903}{7758197257970}f_{n+3} \end{array} \right] \quad (12)$$

respectively.

The method (10) and its derivatives (11) and (12) are in the same order and are  $p = 8$ . Equation (10) error constant is discovered to be  $C_{p+3} \approx -7.0408 \times 10^{-7}$ .

[16] established that the approach is consistent and zero stable, and that it meets the necessary and sufficient conditions for linear multistep procedures to converge.

### 3. Derivation of starting values for the method

Additional closed starting values are required for the sample open discrete method (10) and its derivatives (11, 12). The primary closed predictors of the same order of accuracy as initial values for  $f_{n+i}$ ,  $i = 3, \dots, k$  are aimed for in this study. As described in section 2, the predictor and its first and second derivatives are created using a mix of power series and exponential function as a basic function. We find the following closed methods that are both consistent and zero stable.

$$y_{n+3} = \left[ -\frac{1794937558}{11657769674} y_n + \frac{62812951659}{58288848235} y_{n+\frac{1}{3}} - \frac{108367520868}{58288848235} y_{n+\frac{2}{3}} + \frac{46015768554}{58288848235} y_{n+1} + \right. \\ \left. \frac{155863248996}{58288848235} y_{n+2} - \frac{317782223082}{58288848235} y_{n+\frac{7}{3}} + \frac{228721310766}{58288848235} y_{n+\frac{8}{3}} \right] + \\ \frac{h^3}{11657769647} \left[ 219663738 f_{n+\frac{8}{3}} - 252382088 f_{n+2} + 385123984 f_{n+1} - 4502610 f_n \right] \quad (13)$$

$$hy'_{n+3} = \left[ -\frac{4308576613683}{1305670200464} y_n + \frac{1863061234866}{81604387529} y_{n+\frac{1}{3}} - \frac{19847853948087}{466310785880} y_{n+\frac{2}{3}} + \right. \\ \left. \frac{6064263432347}{233155392940} y_{n+1} - \frac{390128587253}{93262157176} y_{n+2} - \frac{4428856664613}{1632087750580} y_{n+\frac{7}{3}} + \frac{25592991212799}{6528351002320} y_{n+\frac{8}{3}} \right] + \\ h^2 \left[ \frac{78202578303}{466310785880} f_{n+\frac{8}{3}} + \frac{20155338809}{69946617882} f_{n+2} + \frac{106947925682}{174866544705} f_{n+1} - \frac{785685495}{93262157176} f_n \right] \quad (14)$$

$$h^2 y''_{n+3} = \left[ -\frac{1175180958347801}{26113404009280} y_n + \frac{1266394778781507}{4080219376450} y_{n+\frac{1}{3}} - \frac{5482875532126437}{466310785880} y_{n+\frac{2}{3}} + \right. \\ \left. \frac{6064263432347}{233155392940} y_{n+1} - \frac{390128587253}{93262157176} y_{n+2} - \frac{4428856664613}{1632087750580} y_{n+\frac{7}{3}} + \frac{25592991212799}{6528351002320} y_{n+\frac{8}{3}} \right] + \\ h \left[ \frac{78202578303}{466310785880} f_{n+\frac{8}{3}} + \frac{20155338809}{69946617882} f_{n+2} + \frac{106947925682}{174866544705} f_{n+1} - \frac{785685495}{93262157176} f_n \right] \quad (15)$$

The main predictor (13) and its derivatives (14) and (15) are both of order 8 with absolute error constants  $c_{p+3} = 9.7188 \times 10^{-8}$ ,  $C_{p+3} = 2.1802 \times 10^{-6}$  and  $C_{p+3} = 3.0415 \times 10^{-5}$  respectively. Taylor's series expansion was adopted to evaluate  $y_{n+\frac{1}{3}}$  and  $y_{n+\frac{2}{3}}$ .

#### 4. Numerical Experiments

The usability and accuracy of the derived method is confirmed with linear and non-linear third order ordinary differential equations and the results are compared with the results of existing methods.

##### Problem 1

$$y''' + 4y' = x, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1,$$
$$h = 0.1$$

**Theoretical solution:**  $y(x) = \frac{3}{16}(1 - \cos 2x) + \frac{1}{8}x^2$

Table 1 shows the absolute error obtained comparing with an order 9 method in [18] for same values of N (total number of sub intervals TS) and end point of integration, b. the errors in new method demonstrates the efficiency and good accuracy of the method.

##### Problem 2

$$y''' = e^x, \quad y(0) = 3, \quad y'(0) = 1, \quad y''(0) = 5, \quad h = 0.1$$

**Theoretical solution:**  $y(x) = 2 + 2x^2 + e^x$

The errors for this problem are compared with [18], where a hybrid block method of order nine and step number five was presented in Table 2. The basis for comparison is in terms of high order and the step number yet the new method performed better than the method [18].

##### Problem 3

$$y''' = 3\sin x, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2,$$
$$h = 0.1$$

**Theoretical solution:**  $y(x) = 3\cos x + \frac{x^2}{2} - 2$

Table 3 shows the comparison of the maximum errors of the new method and method [19] with high order nine with same step number three. The results show that the new method appreciates more than [19].

#### Problem 4

$$y''' = y'(2xy'' + y'), \quad y(0) = 1, \quad y'(0) = \frac{1}{2},$$

$$y''(0) = 0, \quad h = 0.1$$

$$\text{Theoretical solution: } y(x) = 1 + \frac{1}{2} \ln \left[ \frac{2+x}{2-x} \right].$$

Table 4 shows the error for Problem 4 using the new method and compared with hybrid method [20] with order six and step number eight  $p = 6, k = 8$ .

#### Problem 5: (The Famous Blassius Equation)

$$2y''' + yy'' = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$$

#### No theoretical solution

Table 5 shows the comparison of the computed solution of the new method with that of [20],  $p = 6, k = 8$ , at twelfth decimal point.

#### Problem 6

$$y'''(x) + y'(x) = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2, \quad x \in [0, 1]$$

$$\text{Theoretical solution: } y(x) = 2(1 - \cos x) + \sin x$$

Table 6 shows the numerical solution and the maximum error  $|y_c - y_{ex}|$  of Problem 6 when solved by the new method.

#### Problem 7

$$y''' + 2y'' - y' - 2y = e^x, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 0, \quad h = 0.1$$

$$\text{Theoretical solution: } y(x) = \frac{1}{36} (43e^x + 9e^{-x} - 16e^{-2x} + 6xe^x)$$

Table 7 shows the numerical solution and the maximum error  $|y_c - y_{ex}|$  of Problem 7 when solved by the new method.

#### Problem 8

$$y''' = -6(y)^4, \quad y(1) = -1, \quad y'(1) = -1, \quad y''(1) = -2, \quad h = 0.05$$

$$\text{Theoretical solution: } y(x) = \frac{1}{(x-2)}$$

Table 8 shows the numerical solution, the maximum error  $|y_c - y_{ex}|$  and the time of execution to measure its efficiency of Problem 8 when solved by the new method.

### Problem 9

$$y''' - y'' + y' - y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1,$$
$$h = 0.01.$$

**Theoretical solution:**  $y(x) = \cos x$

Table 9 shows the numerical solution, the maximum error  $|y_c - y_{ex}|$  and the time of execution to measure its efficiency of Problem 9 when solved by the new method.

### Problem 10

$$y''' + y'' + 3y' - 5y = 2 + 6x - 5x^2, \quad y(0) = -1, \quad y'(0) = 1,$$
$$y''(0) = -3$$

**Theoretical solution:**  $y(x) = x^2 - e^x + e^{-x} \sin(2x)$

Table 10 shows the numerical solution, the maximum error  $|y_c - y_{ex}|$  and the time of execution to measure its efficiency of Problem 10 when solved by the new method.

The following notations are used

- 3S4PHM : 3 Step 4 Point Hybrid Method
- b: different end point of integration
- TS: Total number of sub intervals
- $y_{ex}$  : Theoretical solution
- $y_c$  : Numerical solution
- $A_e$  : Maximum error
- $t_e(s)$  : Time of execution in Microseconds

**Table 1.** Numerical results and Comparison of errors for Problem 1  $k = 3, p = 8, h = 0.1$

| $b$ | TS  | $y_{ex}$           | $y_c$              | $A_e$        | Modebe <i>et al.</i> |
|-----|-----|--------------------|--------------------|--------------|----------------------|
|     |     |                    |                    |              | [17], $k = 3, p = 9$ |
| 5   | 30  | 0.0049875166547672 | 0.0049875166547652 | 2.168404e-17 | 2.14e-12             |
|     | 45  | 0.0198010636244590 | 0.0198010636244264 | 7.667478e-16 | 3.79e-14             |
|     | 60  | 0.0439995722044353 | 0.0439995722044285 | 6.862566e-15 | 1.32e-15             |
| 10  | 60  | 0.0768674919974065 | 0.0768674919971374 | 2.063627e-14 | 4.81e-12             |
|     | 75  | 0.1174433176497239 | 0.1174433176497062 | 4.825307e-14 | 5.24e-13             |
|     | 90  | 0.1645579210356238 | 0.1645579210355062 | 1.176559e-13 | 8.52e-14             |
| 15  | 75  | 0.2168811607062050 | 0.2168811607061243 | 2.019773e-13 | 4.69e-11             |
|     | 90  | 0.2729749104314919 | 0.2729749104312486 | 3.265166e-13 | 7.70e-12             |
|     | 105 | 0.3313503927549541 | 0.3313503927543995 | 5.546119e-13 | 1.66e-12             |
| 90  | 90  | 0.3905275318525895 | 0.3905275318525248 | 7.848167e-13 | 1.75e-10             |



**Table 2.** Numerical results and Comparison of errors for Problem 2  $k = 3, p = 8, h = 0.1$

| $x$ | $y_{ex}$           | $y_c$              | $A_e$        | Error in Awoyemi <i>et al.</i> [18], $k = 5, p = 9$ |
|-----|--------------------|--------------------|--------------|-----------------------------------------------------|
| 0.1 | 3.1251709180756477 | 3.1251709180756471 | 4.440892e-16 | 0.0000e-00                                          |
| 0.2 | 3.3014027581601697 | 3.3014027581601239 | 1.376677e-14 | 2.8422e-13                                          |
| 0.3 | 3.5298588075760033 | 3.5298588075760193 | 1.598721e-14 | 1.6729e-12                                          |
| 0.4 | 3.8118246976412706 | 3.8118246976412596 | 1.643130e-14 | 2.9983e-11                                          |
| 0.5 | 4.1487212707001282 | 4.1487212707001253 | 8.881784e-15 | 3.1673e-11                                          |
| 0.6 | 4.5421188003905097 | 4.5421188003904778 | 3.197442e-14 | 9.1899e-11                                          |
| 0.7 | 4.9937527074704775 | 4.9937527074704615 | 2.309264e-14 | 8.9531e-11                                          |
| 0.8 | 5.5055409284924695 | 5.3982722654981785 | 2.486900e-14 | 1.9168e-10                                          |
| 0.9 | 6.0796031111569526 | 6.0796031111569375 | 1.509903e-14 | 2.1111e-10                                          |

1.0 6.7182818284590482 6.7182818284590371 3.019807e-14 4.9398e-10

**Table 3.** Numerical results and Comparison of errors for Problem 3  $k = 3, p = 8, h = 0.1$

| $x$ | $y_{ex}$           | $y_c$              | $A_e$        | Error in Kashkari & Alqarni, [19], $k = 3, p = 9$ |
|-----|--------------------|--------------------|--------------|---------------------------------------------------|
| 0.1 | 0.9900124958340770 | 0.9900124958340766 | 3.774758e-15 | 4.1078e-15                                        |
| 0.2 | 0.9601997335237251 | 0.9601997335237256 | 2.220446e-16 | 1.6875e-14                                        |
| 0.3 | 0.9110094673768181 | 0.9110094673768163 | 1.776357e-15 | 5.0848e-14                                        |
| 0.4 | 0.8431829820086554 | 0.8431829820086507 | 1.665335e-15 | 1.1779e-13                                        |
| 0.5 | 0.7577476856711178 | 0.7577476856711180 | 4.218847e-15 | 2.4081e-13                                        |
| 0.6 | 0.6560068447290344 | 0.6560068447290355 | 1.110223e-15 | 4.3709e-13                                        |
| 0.7 | 0.5395265618534650 | 0.5395265618534664 | 7.105427e-15 | 7.3708e-13                                        |
| 0.8 | 0.4101201280414957 | 0.4101201280414976 | 7.771561e-15 | 1.1662e-12                                        |
| 0.9 | 0.2698299048119925 | 0.2698299048120023 | 9.714451e-15 | 1.7587e-12                                        |
| 1.0 | 0.1209069176044184 | 0.1209069176044767 | 1.507128e-14 | 2.5466e-12                                        |

**Table 4.** Numerical results and Comparison of errors for Problem 4,  $k = 3, p = 8, h = 0.1$

| $x$ | $y_{ex}$             | $y_c$                | $A_e$        | Error in Adoghe & Omole, [20] |
|-----|----------------------|----------------------|--------------|-------------------------------|
| 0.1 | 1.050041729278491400 | 1.050041729369078900 | 9.058754e-11 | 3.3307e-16                    |
| 0.2 | 1.100335347731075800 | 1.100335348536302100 | 8.052263e-10 | 3.6746e-13                    |
| 0.3 | 1.151140435936467000 | 1.151140438940802000 | 3.004335e-10 | 2.6199e-09                    |
| 0.4 | 1.202732554054082300 | 1.202732561923759000 | 7.869677e-09 | 1.7307e-08                    |
| 0.5 | 1.255412811882995700 | 1.255412828889007000 | 1.700601e-09 | 5.8127e-08                    |
| 0.6 | 1.309519604203112100 | 1.309519636789381300 | 3.258627e-09 | 1.4214e-07                    |
| 0.7 | 1.365443754271396400 | 1.365443811831118500 | 5.755972e-08 | 2.8366e-07                    |
| 0.8 | 1.423648930193602200 | 1.423649026147731000 | 9.595413e-08 | 4.9755e-07                    |
| 0.9 | 1.484700278594052200 | 1.484700431913873200 | 1.533198e-08 | 7.9519e-07                    |

1.0 1.549306144334055400 1.549306381725843900 2.373918e-07 1.1822e-06

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**Table 5:** Comparison of Numerical results for Problem 5,  $k = 3, p = 8, h = 0.1$

| x   | $y_c$                | $y_c$ in Adoghe & Omole [20] |
|-----|----------------------|------------------------------|
| 0.1 | 0.004999958333925108 | 0.004999958334               |
| 0.2 | 0.019998666643451606 | 0.019998666843               |
| 0.3 | 0.044989879385560247 | 0.044989878832               |
| 0.4 | 0.079957357138892723 | 0.079957372944               |
| 0.5 | 0.124870055528146960 | 0.124870034341               |
| 0.6 | 0.179676815966278940 | 0.179677059379               |
| 0.7 | 0.244303601485360510 | 0.244303394383               |
| 0.8 | 0.318643838964219660 | 0.318645494176               |
| 0.9 | 0.402568546161517270 | 0.402567554422               |
| 1.0 | 0.495890885929872240 | 0.495898354558               |

**Table 6.** Numerical results for Problem 6,  $k = 3, p = 8, h = 0.1$

| x   | $y_{ex}$           | $y_c$              | $A_e$        |
|-----|--------------------|--------------------|--------------|
| 0.1 | 0.1098250860907768 | 0.1098250860907769 | 1.636780e-16 |
| 0.2 | 0.2385361751125785 | 0.2385361751125786 | 1.942890e-16 |
| 0.3 | 0.3848472284101276 | 0.3848472284101272 | 4.440892e-16 |
| 0.4 | 0.5472963543028797 | 0.5472963543028726 | 1.776357e-15 |
| 0.5 | 0.7242604148234562 | 0.7242604148234573 | 1.665335e-15 |
| 0.6 | 0.9139712435756802 | 0.9139712435756791 | 1.110223e-15 |
| 0.7 | 1.1145333126687111 | 1.1145333126687148 | 7.549517e-15 |
| 0.8 | 1.3239426722051881 | 1.3239426722051825 | 4.440892e-16 |
| 0.9 | 1.5401069730861576 | 1.5401069730861598 | 2.220446e-15 |
| 1.0 | 1.7608663730716114 | 1.7608663730716119 | 1.110223e-15 |

**Table 7.** Numerical results for Problem 7,  $k = 3, p = 8, h = 0.1$

| $x$ | $y_{ex}$           | $y_c$              | $A_e$        |
|-----|--------------------|--------------------|--------------|
| 0.1 | 1.2008137983659488 | 1.2008137983659457 | 4.218847e-15 |
| 0.2 | 1.4063738319947532 | 1.4063738319947469 | 3.441691e-14 |
| 0.3 | 1.6211125663343329 | 1.6211125663343400 | 1.021405e-13 |
| 0.4 | 1.8492349517044138 | 1.8492349517042301 | 2.120526e-13 |
| 0.5 | 2.0948300925243486 | 2.0948300925242456 | 4.507505e-13 |
| 0.6 | 2.3619703731235751 | 2.3619703731235733 | 6.838974e-13 |
| 0.7 | 2.6548012251017616 | 2.6548012251018234 | 9.849899e-13 |
| 0.8 | 2.9776242436411207 | 2.9776242436432563 | 1.461054e-12 |
| 0.9 | 3.3349759807254569 | 3.3349759807235646 | 1.892264e-12 |
| 1.0 | 3.7317044453680599 | 3.7317044453664352 | 2.387424e-12 |

**Table 8.** Numerical results for Problem 8,  $k = 3, p = 8, h = 0.1$

| $x$  | $y_{ex}$            | $y_c$               | $A_e$        | $t_e(s)$ |
|------|---------------------|---------------------|--------------|----------|
| 1.05 | -1.0526315789473684 | -1.0526315779293796 | 1.017989e-09 | 0.029    |
| 1.10 | -1.1111111111111112 | -1.1111110853730750 | 2.573804e-08 | 0.031    |
| 1.15 | -1.1764705882352944 | -1.1764704664689964 | 1.217663e-07 | 0.032    |
| 1.20 | -1.2500000000000002 | -1.2499996375184190 | 3.624816e-07 | 0.032    |
| 1.25 | -1.3333333333333337 | -1.3333324687923525 | 8.645410e-07 | 0.032    |
| 1.30 | -1.4285714285714290 | -1.4285696093065769 | 1.819265e-06 | 0.033    |
| 1.35 | -1.5384615384615392 | -1.5384579871365800 | 3.551325e-06 | 0.033    |
| 1.40 | -1.6666666666666676 | -1.6666600337776776 | 6.632889e-06 | 0.033    |
| 1.45 | -1.8181818181818195 | -1.8181697022691894 | 1.211591e-05 | 0.034    |
| 1.50 | -2.0000000000000018 | -1.9999779695058428 | 2.203049e-05 | 0.034    |

**Table 9.** Numerical results for Problem 9,  $k = 3, p = 8, h = 0.1$

| $x$ | $y_{ex}$             | $y_c$                | $A_e$        | $t_e(s)$ |
|-----|----------------------|----------------------|--------------|----------|
| 0.1 | 0.995004165278025710 | 0.995004165278025720 | 3.885781e-18 | 0.0405   |
| 0.2 | 0.980066577841241630 | 0.980066577841242410 | 6.483702e-16 | 0.0443   |
| 0.3 | 0.955336489125605980 | 0.955336489125600100 | 5.884182e-15 | 0.0695   |
| 0.4 | 0.921060994002885100 | 0.921060994002833000 | 1.329603e-15 | 0.0483   |
| 0.5 | 0.877582561890372650 | 0.877582561886991330 | 3.381406e-15 | 0.0497   |
| 0.6 | 0.825335614909678110 | 0.825335614909678880 | 7.771561e-16 | 0.0754   |
| 0.7 | 0.764842187284488270 | 0.764842187270708270 | 1.378786e-15 | 0.0549   |
| 0.8 | 0.696706709347165170 | 0.696706709323105170 | 2.406098e-15 | 0.0562   |
| 0.9 | 0.621609968270663950 | 0.621609968270670050 | 6.106227e-15 | 0.0807   |
| 1.0 | 0.540302305868139320 | 0.540302305807189320 | 6.095324e-15 | 0.0590   |

**Table 10.** Numerical results for Problem 10,  $k = 3, p = 8, h = 0.1$

| $x$ | $y_{ex}$              | $y_c$                 | $A_e$        | $t_e(s)$ |
|-----|-----------------------|-----------------------|--------------|----------|
| 0.1 | -0.915407473756112420 | -0.915407473756113530 | 1.110223e-15 | 0.0793   |
| 0.2 | -0.862573985499429210 | -0.862573985499429760 | 5.551115e-16 | 0.0950   |
| 0.3 | -0.841561375114168840 | -0.841561375114167620 | 1.221245e-15 | 0.1125   |
| 0.4 | -0.850966529765556200 | -0.850966529765555870 | 3.330669e-16 | 0.1270   |
| 0.5 | -0.88834331915555420  | -0.888343319155558640 | 3.219647e-15 | 0.1447   |
| 0.6 | -0.950604904717254890 | -0.950604904717256780 | 1.887379e-15 | 0.1629   |
| 0.7 | -1.034392853932995000 | -1.034392853932995200 | 2.220446e-16 | 0.1808   |
| 0.8 | -1.136403556878909700 | -1.136403556878907900 | 1.776357e-15 | 0.1957   |
| 0.9 | -1.253666211231613900 | -1.253666211231615300 | 1.332268e-15 | 0.2121   |

|     |                       |                       |              |        |
|-----|-----------------------|-----------------------|--------------|--------|
| 1.0 | -1.383769999219784100 | -1.383769999219783600 | 4.440892e-16 | 0.2279 |
|-----|-----------------------|-----------------------|--------------|--------|

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## 5. Discussion of Results

Problems 1-5 in Modebei *et al.* (2021), Awoyemi *et al.* (2014), Kashkari and Alqarni, (2019), Adoghe and Omole, (2019) were solved with the 3-step approach of order 8, in Modebei *et al.* (2021), Awoyemi *et al.* (2014), Kashkari and Alqarni, (2019) and despite their higher order of accuracy and step number, the results in Tables 1-5 indicate better accuracy than the previous approaches. The novel method was also utilized to solve more linear and nonlinear issues 6-10 of third order initial value ODEs, with the results in Tables 6-10 demonstrating that the method is practical and efficient.

## 6. Conclusion

A predictor-corrector technique was created, with the main predictor having the same order of accuracy as the corrector. The approach was found to be zero stable, consistent with order eight, and with a low error constant during testing. The method's accuracy is improved by developing predictors with the same order of accuracy as the corrector.

The method was used to directly solve both linear and nonlinear third order differential equations problems.

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